DERIVATION OF ADDITIONAL PROBABILISTIC INFORMATION FOR ANALYZING DECISIONS

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THESIS

DERIVATION OF ADDITIONAL PROBABILISTIC INFORMATION FOR ANALYZING DECISIONS UNDER RISK

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Derivation of Additional Probabilistic Information

for

Analyzing Decisions Under Risk

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ABSTRACT

The problem of analyzing decisions under risk is investigated. The vehicle for this investigation is the single-period inventory model commonly referred to as the "newsboy problem." The maximum expected value of the profit is the criteria most often used in making a decision in this type of problem. This paper analyzes this model in several ways. It is found that, under some conditions, variance of profit provides valuable insight into the problem and could assist the analyst in choosing alternatives for the decision maker. In addition, it is found that in this model, maximizing the expected value of profit does not maximize the probability of attaining at least this maximum expected profit. Appropriate display of this additional information allows the decision maker to implicitly assign his utilities in terms of his preferences when making decisions under risk. A modification of the expected utility function used by Borch is also discussed so that, for managers who exhibit decreasing marginal utility for money, some of this additional information can be simplified to produce a single measure of merit.

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I. INTRODUCTION

A. DISCUSSION

Possibly the most common principle used as a criteria for making decisions under risk is to choose the action which maximizes the expected value of profit (or cost or some other measure of merit). Throughout this paper the measure of merit will be profit. Many examples can be offered where this criteria is not satisfactory to a manager who wants information for making a decision, nor is it satisfactory to explain the behavior of decision makers. See Morris, Reference [1] . One feature these examples seem to have in common is that the decisions involved are to be made once, or at most, a few times.

Another feature that they seem to have in common is that two decision makers will often make differing decisions. A good example is insurance; some people insure and some do not. It is clear then that decision makers use different principles in making their choices. It is the analyst's role to be aware of these various criteria and evaluate their application to a particular problem. The aspiration principle - a manager selects from acceptable alternatives rather than from optimal alternatives, according to Cyert and March [2] - appears to be an existing decision principle under risk. Another is the expectation - variance principle (see Morris [1]). This principle, while related to maximum expected value principle, suggests that more information for

decision making can be gained by attaining knowledge of the variance of profit. It may be that the manager has decreasing marginal utility for money. Such a manager is termed a risk averter and, thus, he would view the variance of profit with aversion (see Morris [1]). This additional information may lead to a different decision by the manager than one based solely on maximum expected value.

Another way of looking at decisions under risk is to attempt to construct a utility function for the manager. If this is done, an analytic function could be fit to the manager's constructed utility function and the analyst could then select alternatives based on this explicit statement of the decision maker's attitude toward risk. However, this can result in a more difficult mathematical programming problem or the utility function may be used for ordering alternatives which are not like the experimental alternatives used to construct the utility function.

Quite often these difficulties are avoided in the analysis by choosing the alternative which maximizes expected profit. This is a relatively easy mathematical programming which takes the form

$$\underset{h}{\text{Maximize}} \quad E(P_h) = \int_{-\infty}^{\infty} x\,dG_h(x)$$

where $G_h(x)$ is the cumulative distribution function of profit and is equivalent to taking as a utility function $u_1(x) = x$ and maximizing the expected value of this utility function. The difficulty of the mathematical problem grows

rapidly when the next more sophisticated utility function is used, say $u_2(x) = x - \alpha x^2, \quad \alpha > 0$. The problem is then

$$\underset{h}{\text{Maximize}} \quad E(P_h) - \alpha\left(V(P_h) + \left[E(P_h)\right]^2\right) =$$

$$\int_{-\infty}^{\infty} (x - \alpha x^2) \; d \; G_h \; (x)$$

One choice is then presented to the manager and he either selects it or he doesn't. The use of $u_1(x)$ has its advantages; it reduces the information down to a single measure of merit. The advantage is retained at the cost of a more difficult programming problem when $u_2(x)$ is used. In any case, this scheme has implicitly chosen the decision maker's utility function for him.

If the manager exhibits decreasing marginal utility for money and thus has an aversion for variance of profit, then it may not be necessary to solve a more difficult mathematical programming problem. Various alternatives can be selected based on variance, and, if necessary, utilities can be assigned to these selected alternatives.

This thesis, then, is that the analyst can develop additional probabilistic information which can, at least, sharpen his insight and the insight of the decision maker into the various alternatives available, and, at best, provide the information that will allow a decision maker to implicitly apply other criteria which reflect his attitude toward risk. In addition, the analyst may know something about the

decision maker's utility function and be able to assign utilities to the alternatives.

The medium for this thesis is the single - period inventory model known as the "newsboy problem." Additional probabilistic information is derived and calculated keeping various principles of choice under risk in mind. A modification is then suggested for the mathematical model of the expected utility function for risk averters (see reference [4]). This modified expected utility function is used to compare pairs of selected alternatives.

To complete the prerequisites, the model most commonly used to solve the newsboy problem is now stated and discussed.

B. SINGLE - PERIOD INVENTORY MODEL

1. The Model

The newsboy problem is an example of the single - period inventory model discussed in reference [3] . Assume that the decision maker has to decide how many items of a certain product to buy to meet his demands during a certain period with no choice of re-ordering. He buys an item for \$C and sells it for \$S. If he does not sell an item, he can salvage it for \$L ($L < C$) and if a customer demands an item and the manager is out of stock, the manager incurs a cost of \$$\pi$ for each item out of stock (this cost is in addition to lost profit on the lost sale). Demand is a random variable, X, with a mass or density function $f(x)$ and a cumulative distribution function (cdf), $F(x)$, with a mean μ and a variance σ^2. The decision variable is the number of items

to buy, h. The solution in Hadley and Whitin [3] is to derive an expression for the expected profit, $E(P_h)$, and maximize with respect to h. This leads to a quantity h^* which satisfies, in the continuous case,

$$\bar{F}(h^*) = \frac{C-L}{S-L+\pi} \tag{1}$$

where $\bar{F}(x)$ is the complementary cdf of the demand random variable X. The discrete case is analogous. The function $E(P_h)$ is usually strictly concave so the solution to equation (1) is usually unique. Throughout the remainder of the paper, it will be assumed that demand is normally distributed.

2. <u>Discussion of the Model</u>

Since X is a random variable, so is profit P_h. Profit, P, is indexed by h to indicate that there is a different random variable, profit, for each value of h. The maximum expected value criteria selects the h which maximizes the expected value of profit. This is equivalent to selecting the distribution of profit that has the highest mean. Hillier [5] discusses this problem when there are two risky investments from which to choose. He derives the distribution of return on investment for each of the investments and suggests presenting to the decision maker sketches of the density functions of the random variable, return on investment, rather than just presenting the two numbers which represent the mean return on the investment for each

investment. Such probabilistic information is valuable to the analyst and to the decision maker. The variance of P_h* as compared to other distributions of profit may be important to a decision maker, especially one who is a risk averter. Does P_h* have the smallest variance of all P_h ? Is there another P_h which, even though it has a smaller mean, also has a smaller variance such that the decision maker may prefer it to the one with the highest mean? Does P_h* have the highest probability of attaining the manager's aspiration level, or, more interesting, does it have the highest probability of attaining maximum expected profit?

These then are some of the questions the analyst could ask and their answers may help the decision maker select an alternative more suitable to his preferences.

II. DERIVATION OF ADDITIONAL INFORMATION

A. VARIANCE OF PROFIT

To obtain the variance of profit, $V\left[P_h\right]$, the second moment of P_h, $E\left[P_h^2\right]$, is first derived.

$$E\left[P_h^2\right] = E\left[P_h^2 \mid x \leq h\right] P\left[X \leq h\right] + E\left[P_h^2 \mid x > h\right] P\left[X > h\right] \quad (2)$$

by the law of total probability. Given that demand

$$E\left[P_h^2\right] = \left[Sx - Ch + L\ (h-x)\right]^2$$

since profit is the selling price times the units demanded less the cost of the items procured plus the salvage value of the items left over. Given that demand $x > h$,

$$E\left[P_h^2\right] = \left[Sh - Ch - \pi(x-h)\right]^2$$

since profit is the profit from items procured and sold less the stock out costs for items demanded but not in stock. Returning to equation (2)

$$E\left[P_h^2\right] = \int_0^h \left[Sx - Ch + L(h-x)\right]^2 f(x)\,dx + \int_h^\infty \left[Sh - Ch - \pi(x-h)\right]^2 f(x)\,dx\ .$$

Little error will occur in taking the first integral from zero as long as $\mu \geq 3\sigma$.

In the following form, the $E\left[P_h^2\right]$ is easily coded in FORTRAN IV where subroutines are available to evaluate the ordinate, ϕ, and the complementary cdf, Φ, of the standard normal distribution:

$$E\left[P_h^2\right] = (S-L)^2\,(\sigma^2+\mu^2) - 2(S-L)(C-L)\,h\mu + (C-L)^2h^2$$

$$+\Big(\left[\pi^2-(S-L)^2\right](\sigma^2+\mu^2) - 2(S-C+\pi)\,\pi\,h\mu$$

$$+\,2(S-L)(C-L)h\mu + \left[(S-C+\pi)^2-(C-L)^2\right]h^2\Big)\,\Phi\left(\frac{h-\mu}{\sigma}\right)$$

$$+\Big(\left[\pi^2-(S-L)^2\right]\sigma(h+\mu) - 2\sigma(S-C+\pi)\,\pi\,h$$

$$+\,2\sigma(S-L)(C-L)h\Big)\,\phi\left(\frac{h-\mu}{\sigma}\right).$$

When demand is normal, the expected profit can be written

$$E\left[P_h\right] = (S-L)\mu - (C-L)h + (S-L+\pi)(h-\mu)\Phi\left(\frac{h-\mu}{\sigma}\right)$$

$$-\,\sigma(S-L+\pi)\,\phi\left(\frac{h-\mu}{\sigma}\right).$$

Thus

$$V\left[P_h\right] = E\left[P_h^2\right] - \left(E\left[P_h\right]\right)^2. \qquad (3)$$

As h increases $V\left[P_h\right]$ approaches $(S-L)^2\sigma^2$. The details of the above derivation are contained in Appendix A. The cdf of profit will be derived next.

B. DISTRIBUTION OF PROFIT

Recall that, given demand $x \leq h$, profit, $P_h = Sx - Ch + L(h-x)$ and given that $x > h$, $P_h = (S-C)h - \pi(x-h)$. The event that profit for some h will be less than or equal to some amount y (fixed but arbitrary) is the union of the events that $Sx - Ch + L(h-x)$ is less than or equal to y or that $(S-C)h - \pi(x-h)$ is less than or equal to y. These events are disjoint, thus the probability that profit for some h is less than or equal to y, denoted $G_h(y)$ is the sum of the probabilities of the two events. In symbols,

$$G_h(y) = P\left[Sx - Ch + L(h-x) \leq y\right] + P\left[(S-C)h - \pi(x-h) \leq y\right].$$

This is true for $y < (S-C)h$ since this is the maximum profit possible. For $y \geq (S-C)h$, $G_h(y) = 1$.

Rewriting in terms of the demand distribution,

$$G_h(y) = P\left[X \leq \frac{y+(C-L)h}{(S-L)}\right] + P\left[X > \frac{(S-C+\pi)h-y}{\pi}\right] \qquad y < (S-C)h$$

and in terms of the standard normal random variable,

$$G_h(y) = 1 - \Phi\left[\frac{y+(C-L)h - (S-L)\mu}{(S-L)\sigma}\right] + \Phi\left[\frac{(S-C+\pi)h-y-\pi\mu}{\pi\sigma}\right] \qquad y < (S-C)h$$

$$= 1 \qquad y \geq (S-C)h$$

It will be convenient to denote the arguments of Φ above as $l_1(h)$ and $l_2(h)$ respectively. Note that l_1 and l_2 are linear functions in h with positive slopes.

$G_h(y)$ as written above is continuous, as can be verified. Notice that if $\pi = 0$ and thus the cost of stock out is just the lost profit on a lost sale, the last term would not appear and the cdf would take a jump at the maximum profit, (S-C)h. The value of the jump will be

$\Phi\left(\frac{h-\mu}{\sigma}\right)$ since in this case

$$
\begin{aligned}
G_h(y) &= 1 - \Phi\left[l_1(h)\right] \qquad && y < (S-C)h \\
&= 1 && y \geq (S-C)h
\end{aligned}
$$

and letting $y = (S-C)h$ in the first expression gives

$$1 - \Phi\left(\frac{h-\mu}{\sigma}\right).$$

Recall from equation (1) that h^* was chosen such that

$$\Phi\left(\frac{h^*-\mu}{\sigma}\right) = \frac{C-L}{S-L+\pi}.$$

Thus $\frac{C-L}{S-L+\pi}$ for the h^* solution is the probability of attaining the maximum profit, $(S-C)h^*$.

In addition to the analytic form of the distribution being obtained and certain information taken from it (as will be done below), the cdf or density function could be drawn and presented to the decision maker. This procedure was valuable once alternatives were chosen based on other

considerations. This is valuable additional information at least for the analyst if not for the decision maker and is discussed by Hillier (see reference [5]) who was dealing with, at most, a few discrete distributions.

C. PROBABILITY OF MAKING A PROFIT AT LEAST AS GREAT AS SOME FIXED QUANTITY

What distribution of profit P_h maximizes the probability of making a profit at least as great as z ? A question such as this comes up when the analyst perceives that a manager has some profit he wants to attain which is satisfactory. As mentioned before, Cyert and March (see reference [2]) hypothesize that this is, in fact, how many decisions are made. In symbols then

$$\underset{h}{\text{Maximize}} \quad \bar{G}_h(z) = \Phi\left[l_1(h)\right] - \Phi\left[l_2(h)\right]$$

where z is the decision maker's aspiration level. This expression makes sense as long as $z \leq (S-C)h$ or $h \geq \frac{z}{(S-C)}$ since $(S-C)\,h$ is the maximum possible profit. The function is clearly concave in h when $\frac{C-L}{S-C} < \frac{S-C+\pi}{\pi}$, $\pi > 0$, since these are the slopes of the linear arguments. When these conditions are met, $\bar{G}_h(z)$ will have a maximum. It can be seen that the h which satisfies

$$\frac{d\,\bar{G}_h(z)}{dh} = 0 \Longrightarrow \frac{\phi\left[l_1\ (h)\right]}{\phi\left[l_2\ (h)\right]} = \frac{\pi\ (C-L)}{(S-L)\ (S-C+\pi)} \qquad (4)$$

will maximize the probability of making z dollars profit.

The solution of equation (4) is readily found by searching standard normal tables.

Expressions for variance of profit, distribution of profit, distribution of profit and probability of making a profit as great as some fixed amount are now available. Two examples are now studied to see how these expressions can be used to provide additional information for the decision and how it can be presented to the decision maker.

III. NUMERICAL EXAMPLES

A. FIRST EXAMPLE, $\pi = 0$

The parameters of the first example are: $S = \$.25$, $C = \$.19$, $L = \$.15$, $\mu = 300$, $\sigma = 50$, and $\pi = 0$, as used in Hadley and Whitin (see reference [3]). To maximize expected profit, h^* is chosen to satisfy

$$\Phi\left(\frac{h^* - 300}{50}\right) = \frac{C - L}{S - L + \pi} = .4$$

which implies, from the normal tables, $h^* = 313$.

1. Use of Variance of Profit to Select Alternatives

Using the expression for the $V\left[P_h\right]$, equation (3), the variance of profit was calculated for h^* and other values of h. The results are exhibited in Table 1.

TABLE 1

EXPECTED VALUE, VARIANCE OF PROFIT, AND MAXIMUM PROFIT FOR h AND OTHER SELECTED VALUES OF h

h	$E(P_h)$	$V(P_h)$	MAXIMUM PROFIT
242	\$14.25	1	\$14.52
262	15.10	3	15.72
272	15.44	4	16.32
302	16.03	9	18.12
h^*= 313	16.07	11	18.78
422	13.08	25	25.32

From Table 1, the distribution P_h obtained when $h = 262$ has appeal, for example the mean of P_{262} is less than one dollar smaller than the mean of P_h* but the variance of P_h* is more than three times as large as the variance of P_{262}. Its appeal lies in the fact that it is an alternative

P_{262} that in some sense has more certainty than P_{h^*}. In any case, the analyst's judgment is used to select this alternative.

2. Comparison of Selected Alternatives for Decision Maker Using Distribution of Profit

Table 2 compares the distributions of P_h* and P_{262}. Notice from Table 2 that, in addition to "more certainty" for maximum profit obtained from P_{262}, P_h* has more probability of attaining less than $15.72 than does P_{262}; a characteristic not to be overlooked.

TABLE 2

FIRST EXAMPLE, $\pi = 0$
COMPARISON OF VARIOUS PROFIT DISTRIBUTION
CHARACTERISTICS FOR SELECTED ALTERNATIVES

DISTRIBUTION CHARACTERISTICS	P_h*	P_{262}
Mean	$16.07	$15.10
Variance	11	3
Maximum Profit	$18.78	$15.72
$P\left[P_h < 15.72\right]$	.36	.22
$P\left[15.72 < P_h < 18.78\right]$	.24	.00
$P\left[P_h = 15.72\right]$	.00	.78
$P\left[P_h = 18.78\right]$	.40	.00
$P\left[P_h \geq 15.10\right]$	.69	.81

Admittedly, the judgment of the analyst is required to select alternatives for comparison; however, the judgment will be fortified with at least a cursory knowledge of the

decision maker's preferences and aspirations. Before, the decision of how many items to buy was fairly easy to make. Now, with the additional information and knowledge of the risk involved, the decision is not so easily made. At this point the decision maker is required to implicitly assign his utilities to these choices and make a decision.

B. SECOND EXAMPLE, $\pi > 0$

1. Use of the Variance of Profit

The same example is used here as for the first example except π = $.50. This example is also solved in reference [3]. Table 3 lists the mean and variance of profit and maximum profit for h^* and other selected values of h.

TABLE 3

MEAN AND VARIANCE OF PROFIT AND MAXIMUM PROFIT FOR h^* AND OTHER SELECTED VALUES OF h

h	$E(P_h)$	$V(P_h)$	MAXIMUM PROFIT
325	$11.08	86	$19.50
335	12.32	62	20.10
345	13.19	45	20.70
355	13.74	34	21.30
365	14.04	28	21.90
h^* = 375	14.12	25	22.50
385	14.05	23	23.10
425	13.29	24	25.50
435	12.57	25	26.10

Table 3 shows that P_h* does not have the smallest variance of all distributions of profit; however, the difference is small. This small difference also exists for this example when all parameters are varied including the variance of demand. In the first example, the variance of P_{262} was much smaller than the variance of P_h* for a small decrease in expected value. When $\pi > 0$, this is not the case.

The variance of profit as a function of the decision variable takes on its minimum very close to h and the function is very flat in the region of h^*. Accordingly, the variance of profit does not help in the selection of presentable alternatives.

If the analyst perceives that the decision maker may be guided by the aspiration level principle, he can now turn to using the expressions developed in Section II, C, for the probability of making a profit at least as great as some fixed quantity.

2. Use of Probability of Making at Least as Great a Profit as Some Fixed Quantity

The value of this information is revealed by solving equation (4) to see if h^* maximizes the probability of attaining at least the maximum expected profit $E(P_h*) =$ \$14.12. Table 4 contains these values.

TABLE 4

SECOND EXAMPLE
PROBABILITY OF MAKING AT LEAST $E(P_{h^*})$
FOR SELECTED VALUES OF h

h	$\overline{G}_h$ (\$14.12)	$E(P_h)$
355	.5586	\$13.14
356	.5590	
357	.5592	
358	.5593	
359	.5593	13.74
360	.5592	13.89
365	.5572	14.04
$h^* = 375$	.5464	14.12

Table 4 reveals that $h = 359$ gives a higher probability for maximum expected profit than the h which implies that $E(P_h)$ is maximum. If this amount (359) were bought, the expected profit would be \$13.74 but here $P\left[P_{359} \geq 14.12\right] > P\left[P_{h^*} \geq 14.12\right]$. The fact that the differences of probabilities are small is not important. If the decision maker's aspiration level was \$14.12, then P_{359} may be optimal for him. Not knowing explicitly what his aspiration level is, a table could be constructed giving the value of h which maximizes the probability of making a fixed amount for various values of possible aspiration levels. That is, solve equation (4) for appropriately selected values of z.

It is desirable to attempt to simplify some of this information and assign single measure of merit to various alternatives selected.

IV. ASSIGNING A MEASURE OF MERIT TO SELECTED ALTERNATIVES

The objection could be raised that some of this additional information may confuse a manager rather than assist him in making a decision. This objection is acknowledged and an attempt is made to deal with it. But another objection also has to be acknowledged. This objection arises in implicitly presenting to the decision maker the expected value of profit as a certain quantity with no indication of the risk. If it is not presented this way, then, to really explain what it means, requires the discussion of variance of some measure of how much the actual profit realization may differ from the mean. In either case, is this not a source of possible confusion also?

A. BORCH'S MODEL

Consider the class of decision makers who exhibit decreasing marginal utility for money. There are various schemes for constructing a utility function of such decision makers. See reference [1] and [6]. For this class of decision makers, the utility function could be fitted with a parabola of the form $x - \alpha x^2$, $\alpha > 0$ for payoffs $x \leq 1/2\alpha$. (This restriction is required since for $x > 1/2\alpha$ the slope of the parabola is negative which would violate common sense notions of utility.) This utility function is studied by Borch [4]. Markowitz [7] has shown that proper use of the expected value and variance of profit as suggested above is equivalent to maximizing the expected value of such

a utility function and he gives justification for using the expected value along with the variance as decision parameters.

A manager whose preferences imply such a utility function exhibits an aversion to the variance of distributions which the utility function is to order. This aversion is measured by the value of α . To be specific, the expected value of $x - \alpha x^2$ over the profit distribution is

$$E(P_h) - \quad E\,(P_h{}^2) = E(P_h) - \alpha V(P_h) - \alpha \left[E(P_h) \right]^2$$

so that it is the second moment, $E\,(P_h{}^2)$, which is the measure of the variance used to order the profit distributions rather than the variance of profit, $V\,(P_h)$. There is no way to write a separate function which will result in expected utility being $E(P_h) - \alpha V(P_h)$. See reference [4] for further reasons for this choice and some examples.

B. MODIFICATION OF BORCH'S MODEL

With the expected utility function in this form, the implication is that all the variance of a given distribution is "bad." Morris [1] arrives at this conclusion when he says that a manager who has such a utility function when presented with two distributions with the same mean would prefer the distribution with the smaller variance.

1. Motivational Example

A possible counter example to the above "principle" is considered. This example also demonstrates the value of more explicit analysis of decisions under risk. Consider

two alternatives where, in the first, profit has a gamma distribution, (P_1), with a mean of 2 and variance of 2; and, in the second, where profit (P_2), has a normal distribution with a mean of 2 and a variance of 1. The principle that one selects the alternative that has smaller variance when the means are the same suggests that one select alternative P_2 if one exhibits decreasing marginal utility for money.

Using Borch's expected utility function to order the two alternatives, the expected utility for P_1 is

$$E(P_1) - \alpha\left(V(P_1) + \left[E(P_1)\right]^2\right) = 2 - 6\alpha$$

whereas the expected utility of P_2 is

$$E(P_2) - \alpha\left(V(P_2) + \left[E(P_2)\right]^2\right) = 2 - 5\alpha$$

which is clearly larger for any $\alpha > 0$, agreeing with the above principle of choosing P_2.

Further analysis shows that the above result and "principle" may not validly order the decision maker's preferences properly even though he exhibits decreasing marginal utility for money. Tables 5, 6, and 7 present the values of $P\left[a \leq P_i \leq b\right]$, $\int_a^b x_i g_i(x_i)\,dx_i$ and $\int_a^b x_i^2\, g_i(x_i)\,dx_i$ for $i = 1, 2$ where g_i is the appropriate density function for profit alternative P_i. These

quantities are defined as fractional probabilities, fractional means and fractional second moments, respectively.

TABLE 5

FRACTIONAL PROBABILITIES FOR PROFIT DISTRIBUTIONS

(a,b)	(-∞,0)	(0,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,∞)	
P_1	.000	.264	.330	.207	.108	.051	.040	1
P_2	.023	.136	.341	.341	.136	.022	.001	1

TABLE 6

FRACTIONAL MEANS FOR PROFIT DISTRIBUTIONS

(a,b)	(-∞,0)	(0,1)	(1,2)	(2,3)	(3,4)	(4,5)	(5,∞)	$E(P_i)$
P_1	.000	.165	.490	.500	.470	.130	.245	2
P_2	-.008	.083	.526	.839	.460	.093	.007	2

TABLE 7

FRACTIONAL SECOND MOMENTS FOR PROFIT DISTRIBUTIONS

(a,b)	(-∞,0)	(0.1)	(1,2)	(2,3)	(3,4)	(4.5)	(5,∞)	$E(P_i^2)$
P_1	.000	.110	.750	1.260	1.280	1.010	1.590	6
P_2	.114	.436	1.151	1.778	1.190	.302	.029	5

As an example to help understand the meaning of the above tables, consider Table 6.

$$E\,(P_2) = E\left[\,(P_2)\,\middle|\, -\infty \leq P_2 \leq 0\right]\; P\left[-\infty \leq P_2 \leq 0\right]$$

$$+\, E\left[\,(P_2)\,\middle|\, 0 \leq P_2 \leq 1\right]\; P\left[0 \leq P_2 \leq 1\right] + \ldots$$

$$+\, E\left[\,(P_2)\,\middle|\, 5 \leq P_2 \leq \infty\right]\; P\left[5 \leq P_2 \leq \infty\right]$$

$$E\,(P_2) = -.008 + .083 + \ldots + .007 = 2$$

The implications of Tables 5, 6, and 7 when making decisions under risk are apparent when one considers Table 7. For P_1, 86% of $E(P_1{}^2)$ lies to the right of $E(P_1)$ whereas only 66% of $E(P_2{}^2)$ lies to the right of $E(P_2)$. Motivated by the idea, that the measure of dispersion of a distribution which measures the possibility of making profits that are larger than what is expected, is "good," this measure is termed "good variance." For the analogous reason, the remainder of this measure is termed "bad variance." Then "good variance" is

$$\int_{\mu_i}^{\infty} x_i{}^2\, g_i\,(x_i)\, dx_i$$

and "bad variance" is

$$\int_{\mu_i}^{\infty} x_i{}^2\, g_i\,(x_i)\, dx_i \quad .$$

These definitions are dependent on the measure of merit. If cost were the measure of merit, the definitions would be reversed.

With these ideas in mind, consider the above tables and note that in Table 5 for P_1 there is no probability of losing money (negative profit) while P_2 offers a .023 probability of losing money. Is there really any risk associated with making large amounts of profit? Is P_1 defaulted when 85% of its second moment is larger than what one expects for a payoff? Is it completely obvious that, even though one has an aversion to risk, one should choose P_2 because it has smaller variance? It is expected that P_1 would be preferred to P_2 by many, if not all, risk averters.

A proposal to resolve this difficulty by modifying somewhat Borch's model for the expected utility function for a manager who exhibits decreased marginal utility for money is made in the next section.

2. <u>The Modification</u>

Let $\alpha = \alpha_1 - \alpha_2$, $\alpha_1 > \alpha_2$, $\alpha_1 > 0$, $\alpha_2 > 0$, where α_1 can be thought of as a measure of how much a manager dislikes losses while α_2 is how much he likes gains. Since $\alpha_1 > \alpha_2$, $\alpha > 0$ is still the measure of risk aversion in the model. The expected utility function now becomes

$$E\left[P\right] - \alpha_1 \int_{-\infty}^{\mu} x^2 \, dG(x) + \alpha_2 \int_{\mu}^{\infty} x^2 \, dG(x)$$

as it will be used to order pairs of alternatives. A simple calculation using Tables 6 and 7 shows, for any feasible

α_1/α_2 , $\alpha_1 > \alpha_2$, that P_1 would be ordered as preferred to P_2.

It should be noted that a utility function could not be written which would result in expected utility as stated above. Therefore, the usefulness of the above is found in comparing alternatives with known mean and variance. It will not help one choose alternatives or could not be used in a mathematical programming problem to maximuze expected utility. It will assist, however, in ordering those alternatives selected by such methods suggested above and elsewhere.

Note that, in the comparison of the gamma distribution and normal distribution, the appeal of the gamma distribution lies in its skewness in the direction of higher profits. This results in the larger percentage of "good variance." In this way, the use of the expected utility measure suggested here takes into account skewness of a distribution. Borch points out [8] that an obvious limitation of the method of ordering probability distributions with a utility function of the form $x - \alpha x^2$, $\alpha > 0$, is that it takes into consideration only the first two moments of the distribution. Suggested herein is a way to practically account for skewness of a distribution (which third moment measures) when ordering sets of distributions.

The goal of this chapter is to attempt to assign a single measure of merit to the alternatives selected above. To demonstrate how this could be done, the first example,

in which the variance of profit was important, is now considered.

3. <u>Use of Modified Expected Utility Function in the First Example</u>

In the first example it was suggested that a distribution which had a smaller variance may have some appeal to some managers. This is reasonable if the manager has preferences which exhibit decreasing marginal utility for money. If so, then the modified expected utility function may help to reduce this to a single measure of merit. Thus, the second moments and bad variance were computed for the two alternatives selected in the first example and are recorded in Table 8.

TABLE 8

COMPARISON OF SECOND MOMENTS AND BAD VARIANCE FOR SELECTED ALTERNATIVES

	$h^*=313$	$h=262$	Feasible α in Borch's Model
Total $E(P_h^2)$	269	231	$1/40 < \alpha < 1/38$
Bad Variance	65	30	

The derivation needed for the above calculations is shown in Appendix B.

The feasible range for α in Table 8 is arrived at by noting that α can be no bigger than $1/2x$ where x is largest profit that $x - \alpha x^2$ is to order. This is about \$19.00. The lower bound on α is the smallest α such that P_{262} is ordered as preferred to $P_h{}^*$. The lower

bound is not zero because in using the expected utility function $E(P_h) - \alpha E(P_h^2)$, P_h^* is taken only as 269/231 more risky than P_{262}. But in terms of getting less profit than expected (when what is expected differs so little) P_h^* is measured more than twice as risky (65/30). It should be noted that good variance was defined in terms of the mean of the given distribution. In section IV, B, above, the means of the distributions being compared were the same. Here they are not and thus the good variance is a measure of getting more profit than expected when you pick a particular distribution with its associated mean. Since the expected utility function contains the mean, the difference in mean will order the distribution properly and the good and bad variance will order the distribution according to how much the distribution deviates from what is expected.

Thus the modified expected utility measure will always order the distribution P_{262} as preferred to P_h^* no matter what feasible α_1 and α_2 are chosen. If $\alpha = 1/40$ and the manager dislikes losses say twice as much as liking gains (i.e. $\alpha_1 / \alpha_2 = 2$) then $\alpha_1 = 2/40$ and $\alpha_2 = 1/40$. Thus $E\,U(P_h^*) = 17.9$ and $E\,U(P_{262}) = 18.6$. This agrees with the knowledge gained by looking at Table 2 as to how much more risky P_h^* is than P_{262}.

V. SUMMARY AND CONCLUSIONS

For the single-period inventory problem, the variance of profit, the distribution of profit, and an expression which usually yields the order quantity which maximizes the probability of making a given profit have been derived when demand is normal. There is no difficulty in doing the same derivations when demand is other than normally distributed. It was seen that this additional information gives the analyst and the decision maker valuable insight into the problem by explicitly displaying the risks involved and allows the decision maker to implicitly assign his preferences based on his total knowledge of his environment and thus his attitudes toward risk.

A way was suggested that, if necessary, given more knowledge of the attitude of some managers toward risk, measures of merit can be practically assigned which will order selected alternative profit distributions.

These specific results may not be applicable to all decisions under risk but it was seen that additional probabilistic information of this kind may result in a decision more appropriate to the decision maker's attitude toward risk than maximizing expected profit. Much of this information was obtained without the existence of a utility function and without a difficult mathematical programming solution.

The probabilistic information derived in any given problem has to be guided by the nature of the problem and

the questions posed by the analyst and manager rather than by the use of the specific results derived here.

APPENDIX A - DERIVATION OF VARIANCE OF PROFIT

From the discussion in section II, A, the second moment $E(P^2)$ is

$$E(P^2) = \int_0^h \left[(S-L)\,x - (C-L)\,h \right]^2 f(x)\,dx + \int_h^\infty \left[(S-C+\pi)\,h - \pi x \right]^2 f(x)\,dx$$

where the subscript h is dropped since here it is clear that $E(P^2)$ and $E(P)$ are functions of h. After carrying out some algebra

$$E(P^2) = (S-L)^2 \int_0^h x^2\, f(x)\,dx - 2(S-L)(C-L)\,h \int_0^h x f(x)\,dx$$
$$+ (C-L)^2 h^2 \int_0^h f(x)\,dx + (S-C+\pi)^2 h^2 \int_h^\infty f(x)\,dx$$
$$- 2(S-C+\pi) \int_h^\infty x\, f(x)\,dx$$
$$+ \pi^2 \int_h^\infty x^2\, f(x)\,dx.$$

Rewriting the first three integrals and carrying out some algebra,

$$E(P^2) = (S-L)^2 (\sigma^2 + \mu^2) - 2(S-L)(C-L)\,h\mu + (C-L)^2 h^2$$
$$+ \left[\pi^2 - (S-L)^2 \right] \int_h^\infty x^2\, f(x)\,dx$$

$$+ \left[2(S-L)(C-L)\ h - 2\ (S-C+\pi)\ \pi\ h \right] \int_h^\infty x\ f(x)\,dx$$

$$+ \left[(S-C+\pi)^2\ h^2 - (C-L)^2\ h^2 \right] \int_h^\infty f(x)\ dx$$

Using equations (1), (3) and (24) in Appendix 4 of reference 3 where $\phi(x)$ and $\Phi(x)$ are the density function and complementary cdf of the standard normal distribution

$$\int_h^\infty x^2\ f(x)\ dx = (\sigma^2 + \mu^2)\ \Phi\left(\frac{h-\mu}{\sigma}\right) + \sigma\ (h+\mu)\ \phi\left(\frac{h-\mu}{\sigma}\right)$$

$$\int_h^\infty xf(x)\ dx = \sigma\ \phi\left(\frac{h-\mu}{\sigma}\right) + \mu\ \Phi\left(\frac{h-\mu}{\sigma}\right)$$

and

$$\int_h^\infty f(x)\ dx = \Phi\left(\frac{h-\mu}{\sigma}\right)$$

are substituted to yield after regrouping

$$E(P^2) = (S-L)^2\ (\sigma^2+\mu^2) - 2\ (S-L)(C-L)\ h\mu + (C-L)^2\ h^2$$

$$+ \Big[\ (\pi^2 - (S-L)^2)(\sigma^2+\mu^2) - 2\ (S-C+\pi)\ \pi\ h\mu$$

$$+ 2(S-L)(C-L)\ h\mu + (S-C+\pi)^2\ h^2 - (C-L)^2\ h^2 \Big]\ \Phi\left(\frac{h-\mu}{\sigma}\right)$$

$$+ \Big[\ (\pi^2 - (S-L)^2)\ \sigma(h+\mu) - 2\sigma(S-C+\pi)\pi h$$

$$+ 2\sigma(S-L)(C-L)\ h \Big]\ \phi\left(\frac{h-\mu}{\sigma}\right).$$

The above expression is easily coded in Fortran IV.

APPENDIX B - DERIVATION OF EXPRESSIONS FOR CALCULATING FRACTIONAL SECOND MOMENT

From section II, C.

$$G_h(y) = 1 - \Phi\left[\frac{y + (C-L)h - (S-L)\mu}{(S-L)\sigma}\right] \qquad y < (S-C)h$$

$$= 1 \qquad y \geq (S-C)h$$

and

$$dG_h(y) = \frac{1}{(S-L)\sigma}\,\phi\left[\frac{y + (C-L)h - (S-L)\mu}{(S-L)\sigma}\right]dy \qquad y < (S-C)h$$

$$= \Phi\left[\frac{h-\mu}{\sigma}\right] \qquad y = (S-C)h$$

$$= 0 \qquad \text{otherwise.}$$

The expected value of utility is

$$EU(P_h) = E(P_h) - {}_1\int_{-\infty}^{E(P_h)} y^2 dG_h(y) + {}_2\int_{E(P_h)}^{\infty} y^2\, dG_h(y).$$

It is only necessary to evaluate the first integral since $E(P_h{}^2)$ to the right of $E(P_h)$ can be obtained by subtraction from entire second moment calculated elsewhere. So

$$\int_{-\infty}^{E(P_h)} y^2\, dG_h(y) = \frac{1}{(S-L)\sigma}\int_{-\infty}^{E(P_h)} y^2\,\phi\left[\frac{y + (C-L)h - (S-L)\mu}{(S-L)\sigma}\right] dy$$

since $E(P_h) < (S-C)h$.

Let

$$z = \frac{y + (C-L)h - (S-L)\mu}{(S-L)\sigma}$$

and

$$r = \frac{E(P_h) + (C-L)h - (S-L)\mu}{(S-L)\sigma}$$

Then

$$\frac{1}{(S-L)\sigma} \int_{-\infty}^{E(P_h)} y^2 \phi \left[\frac{y + (C-L)h - (S-L)\mu}{(S-L)\sigma} \right] dy =$$

$$\int_{-\infty}^{r} \Big[(S-L)^2\sigma^2 z^2 - 2(S-L)(C-L)\ hz + 2(S-L)^2\mu\,\sigma\, z +$$

$$(C-L)^2\ h^2 - 2(S-L)(C-L)h\mu + (S-L)^2\mu^2 \Big] \phi(z)\ dz$$

After carrying out the indicated algebra and making the following substitutions from Appendix 4 of reference 3

$$\int_{-\infty}^{r} z^2 \phi(z)\ dz = 1 - \int_{r}^{\infty} z^2 \phi(z)\ dz = 1 - \Phi(r) - r\,\phi(r)$$

and

$$\int_{-\infty}^{r} z\,\phi(z)\ dz = 0 - \int_{r}^{\infty} z\,\phi(z)\ dz = -\phi(r)$$

and of course

$$\int_{-\infty}^{r} \phi(z)\ dz = 1 - \int_{r}^{\infty} \phi(z)\ dz = 1 - \Phi(r)$$

the resulting expression is easily coded in Fortran IV.

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13. ABSTRACT

The problem of analyzing decisions under risk is investigated. The vehicle for this investigation is the single-period inventory model commonly referred to as the "newsboy problem." The maximum expected value of the profit is the criteria most often used in making a decision in this type of problem. This paper analyzes this model in several ways. It is found that, under some conditions, variance of profit provides valuable insight into the problem and could assist the analyst in choosing alternatives for the decision maker. In addition, it is found that in this model, maximizing the expected value of profit does not maximize the probability of attaining at least this maximum expected profit. Appropriate display of this additional information allows the decision maker to implicitly assign his utilities in terms of his preferences when making decisions under risk. A modification of the expected utility function used by Borch is also discussed so that, for managers who exhibit decreasing marginal utility for money, some of this additional information can be simplified to produce a single measure of merit.

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